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COMMENT

A note on the potential of a homogeneous ellipsoid in ellipsoidal coordinates

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Abstract. Expressions for the interior and exterior potentials of a homogeneous ellipsoid are given in terms of ellipsoidal coordinates and Lamé polynomials. It is shown that Wang's corresponding solutions for spheroidal bodies may be considered as special cases of these general triaxial expressions. Wang's results are also readily obtained by employing the relationship between the Lamé and the Legendre polynomials in the case where the ellipsoid is degenerated into an oblate or prolate spheroid.

1. Introduction

Recently Wang (1988, 1989) has calculated the gravitational potential for an astronomical body of spheroidal shape and obtained analytical expressions in terms of spheroidal coordinate system. Different expressions were derived for the case in which the field point was an exterior (Wang 1988) or interior (Wang 1989) point to the spheroid. A distinction between prolate and oblate spheroid was also made and various expressions were found for each case. Even though the Cartesian analogues of these solutions are given in classical treatise of potential theory (e.g. Kellog 1929, McMillan 1958), the new expressions in terms of spheroidal coordinates may be found useful in directly determining the equipotential surfaces for spheroidal bodies. Wang's analysis is based on volume integration of the expansion of $1/R$ (the inverse of the distance between two points) expressed in spheroidal harmonics.

It is well known that astronomical bodies are generally treated as ellipsoidal masses rather than spheroids and that the triaxial ellipsoidal coordinate system is the most general triply-orthogonal system which yields a separable solution of the Laplace equation (Morse and Feshbach 1953). The purpose of this comment is therefore to extend Wang's solution to general ellipsoidal geometries and to present an alternative simpler method for obtaining such solutions. Thus, instead of employing the corresponding expression for $1/R$ in ellipsoidal harmonics (Miloh 1973a, b) and following a similar procedure to Wang's, it is simpler to start with the Cartesian representation of the potential as demonstrated in the following.

2. Analysis

The general transformation between the orthogonal ellipsoidal coordinates (ρ, μ, ν)

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and the Cartesian system (x_1, x_2, x_3) is (Hobson 1955)

$$
\begin{aligned}
x_1^2 &= \frac{\rho^2\mu^2\nu^2}{h^2k^2} \\
x_2^2 &= \frac{(\rho^2-\mu^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k^2-h^2)} \\
x_3^2 &= \frac{(\rho^2-k^2)(k^2-\mu^2)(k^2-\nu^2)}{k^2(k^2-h^2)}
\end{aligned}
\tag{1}
$$

where

$$k^2 = a_1^2 - a_3^2 \qquad h^2 = a_1^2 - a_2^2 \tag{2}$$

and $a_1 > a_2 > a_3$ denote the three semi-axes of the ellipsoid. The three surfaces, $\rho = \text{constant}$ $(k \leqslant \rho \leqslant \infty)$, $\mu = \text{constant}$ $(h \leqslant \mu \leqslant k)$ and $\nu = \text{constant}$ $(0 \leqslant \nu \leqslant h)$, represent ellipsoids and hyperboloids of one and two sheets respectively.

Following Kellog (1929), the potential induced by a homogeneous ellipsoid of uniform density at a general exterior point $P(\rho, \mu, \nu)$, where $\rho > a_1$, may be written as

$$V_e(\rho, \mu, \nu) = \tfrac{3}{2}M\left(A(\rho) - \sum_{i=1}^{3} B_i(\rho)x_i^2\right) \tag{3}$$

where M is the mass of the ellipsoid,

$$A(\rho) = \int_\rho^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-h^2)(t^2-k^2)}} \tag{4}$$

and

$$B_i(\rho) = \int_\rho^\infty \frac{\mathrm{d}t}{(t^2-a_1^2+a_i^2)\sqrt{(t^2-h^2)(t^2-k^2)}} \qquad i = 1, 2, 3. \tag{5}$$

In a similar manner, the potential at an interior point $P(\rho, \mu, \nu)$, i.e. $\rho < a_1$, is given by

$$V_i(\rho, \mu, \nu) = \tfrac{3}{2}M\left(A(a_1) - \sum_{i=1}^{3} B_i(a_1)x_i^2\right). \tag{6}$$

Thus, given an arbitrary point $P(\rho, \mu, \nu)$, the corresponding potentials are simply found by substituting (1), (4) and (5) into (3) (exterior point) or into (6) (interior point). The resulting potentials are then given in terms of ellipsoidal coordinates. The various integrals in (4) and (5) may be expressed in terms of the Lamé function of the second kind $F_n^m(\lambda)$, which is defined by (Hobson 1955), e.g.,

$$F_n^m(\lambda) = (2n+1)E_n^m(\lambda)\int_\lambda^\infty \frac{\mathrm{d}t}{\{E_n^m(t)\}^2\sqrt{(t^2-h^2)(t^2-k^2)}} \tag{7}$$

where $E_n^m(\lambda)$ denotes the Lamé polynomial of order n and type m. Hence, from (4), (5) and (7), one gets

$$A(\rho) = F_0(\rho) \qquad B_i(\rho) = \tfrac{1}{3}F_1^i(\rho)/E_1^i(\rho) \qquad i = 1, 2, 3 \tag{8}$$

since

$$E_0(\rho) = 1 \qquad E_1^i(\rho) = \sqrt{\rho^2 - a_1^2 + a_i^2} \qquad i = 1, 2, 3. \tag{9}$$

The integrals in (4) and (5) may be also expressed in terms of tabulated elliptic integrals (Miloh 1973b).

3. Applications to spheroids

It is interesting to note how these triaxial general expressions are reduced for the case of a spheroidal body. Thus, for a prolate spheroid $a_1 > a_2 = a_3$ the exterior potential (3) renders for $h = k$

$$V_e(\xi, \mu) = \frac{3}{2}\frac{M}{c}(Q_0(\xi) - \mu^2 \xi Q_1(\xi) + \tfrac{1}{2}(1-\mu^2)(\xi^2-1)^{1/2} Q_1^1(\xi))$$

$$= \frac{M}{c}[Q_0(\xi) - P_2(\mu) Q_2(\xi)] \tag{10}$$

where the spheroidal coordinates (ξ, μ) are related to the cylindrical coordinates (x, r) by

$$x = c\mu\xi \qquad r = c(1-\mu^2)^{1/2}(\xi^2-1)^{1/2} \tag{11}$$

with c denoting the distance between the two foci. Here P_n^m and Q_n^m represent the Legendre polynomials of the first and second kind respectively, where, following Hobson (1955),

$$F_0(\lambda) = Q_0(\lambda) \qquad F_1^1(\lambda) = 3Q_1(\lambda) \qquad F_1^2(\lambda) = F_1^3(\lambda) = -\tfrac{3}{2} Q_1^1(\lambda). \tag{12}$$

Equation (10) has been also derived by Wang (1988, equation (21)) by using a different method and a similar expression for an oblate spheroid may be obtained from (10) by simply replacing $Q_m(\xi)$ by $\mathrm{i}Q_m(\mathrm{i}\xi)$.

Using (6) and (12) the potential at an interior point of a homogeneous prolate spheroid $\xi = \xi_0$ is

$$V_i(\xi, \mu) = \frac{3}{2}\frac{M}{c}\left(Q_0(\xi_0) - \mu^2 \xi^2 \frac{Q_1(\xi_0)}{\xi_0} + \tfrac{1}{2}(1-\mu^2)(\xi^2-1)\frac{Q_1^1(\xi_0)}{(\xi_0^2-1)^{1/2}}\right) \tag{13}$$

which is equivalent to Wang's (1989, equation (7)) expression for the interior potential. Note the complete symmetry between the exterior (10) and the interior (13) potentials, which is not manifested in Wang's solution. Clearly, on the surface of the spheroid $\xi = \xi_0$

$$V_i(\xi_0, \mu) = V_e(\xi_0, \mu) = \frac{M}{c}[Q_0(\xi_0) - P_2(\mu) Q_2(\xi_0)] \tag{14}$$

which may be readily verified from the recurrence formulae between the Legendre polynomials. The internal potential for an oblate spheroid is again obtained from (13) by simply replacing $Q_m(\xi)$ by $\mathrm{i}Q_m(\mathrm{i}\xi)$. Finally, we note that (3) and (6) may be also derived, *à la* Wang, from the volume integration of the corresponding expansion of $1/R$ in ellipsoidal harmonics (Miloh 1973a).

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